



TITLE:

# Duality in Stochastic Optimal Control and Applications (Viscosity Solution Theory of Differential Equations and its Developments)

AUTHOR(S):

Mikami, Toshio; Thieullen, Michele

---

CITATION:

Mikami, Toshio ...[et al]. Duality in Stochastic Optimal Control and Applications (Viscosity Solution Theory of Differential Equations and its Developments). 数理解析研究所講究録 2005, 1428: 106-119

ISSUE DATE:

2005-04

URL:

<http://hdl.handle.net/2433/47319>

RIGHT:

## Duality in Stochastic Optimal Control and Applications

Toshio Mikami\*      Michèle Thieullen†  
Hokkaido University      Université Paris VI

October 29, 2004

### Abstract

We review a duality result and its applications for a stochastic control problem with fixed marginals obtained in [10]. This problem is the stochastic analog of the well known Monge and Monge-Kantorovich optimal transportation problems.

**Keywords:** optimal transportation problem, Legendre transform, duality theorem, stochastic control, forward-backward stochastic differential equation

**Acknowledgements:** the results described below have been presented in a talk at the "RIMS Symposium on Viscosity Solution Theory of Differential Equations and its Developments" July 12-14, 2004. The second author (M. Thieullen) would like to thank the organizers of this symposium (Profs. Y. Giga, H. Ishii, S. Koike) for the opportunity to give this talk and for their very nice welcome.

---

\*Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan; mikami@math.sci.hokudai.ac.jp; phone no. 81/11/706/3444; fax no. 81/11/727/3705; Partially supported by the Grant-in-Aid for Scientific Research, No. 15340047, 15340051 and 16654031, JSPS.

†Corresponding author, Laboratoire de Probabilités et Modèles Aléatoires, Boite 188, Université Paris VI, 75252 Paris, France

## 1 Introduction.

In the present paper we review a duality result and its applications for a stochastic control problem with fixed marginals published in [10]. For a few proofs we do not give all details, rather we preferred to focus on the arguments; details for these proofs can be found in [10].

The problem we are interested in is defined as follows: given  $\epsilon > 0$ ,

$$V_\epsilon(P_0, P_1) := \inf \left\{ E \left[ \int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] \right. \\ \left. PX(t)^{-1} = P_t(t=0, 1), X \in \mathcal{A} \right\}. \quad (1.1)$$

where  $P_0$  and  $P_1$  are Borel probability measures on  $\mathbf{R}^d$  and  $L(t, x; u) : [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto [0, \infty)$  is measurable and convex w.r.t.  $u$ . The infimum is taken over the set  $\mathcal{A}$  of all  $\mathbf{R}^d$ -valued, continuous semimartingales  $\{X(t)\}_{0 \leq t \leq 1}$  on a probability space  $(\Omega_X, \mathbf{B}_X, P_X)$  such that there exists a Borel measurable  $\beta_X : [0, 1] \times C([0, 1]) \mapsto \mathbf{R}^d$  for which

- (i)  $\omega \mapsto \beta_X(t, \omega)$  is  $\mathcal{B}(C([0, t]))_+$ -measurable for all  $t \in [0, 1]$ , where  $\mathcal{B}(C([0, t]))$  denotes the Borel  $\sigma$ -field of  $C([0, t])$ ,
- (ii)  $\{X(t) - X(0) - \int_0^t \beta_X(s, X) ds := \sqrt{\epsilon} W_X(t)\}_{0 \leq t \leq 1}$  where  $W_X$  is a  $\sigma[X(s) : 0 \leq s \leq t]$ -Brownian motion (see [7]).

**Remark** It would appear more natural to consider semi martingales of the form

$$X^u(t) = X_0 + \int_0^t u(s) ds + W(t) \quad (t \in [0, 1]). \quad (1.2)$$

with  $\{u(t)\}_{0 \leq t \leq 1}$  a  $(\mathbf{B}_t)$ -progressively measurable stochastic process. However, if we set

$$\beta_{X^u}(t, X^u) = E[u(t) | X^u(s), 0 \leq s \leq t], \quad (1.3)$$

then using conditional expectations Jensen inequality and convexity of  $L$  one obtains,

$$E \left[ \int_0^1 L(t, X^u(t); u(t)) dt \right] \geq E \left[ \int_0^1 L(t, X^u(t); \beta_{X^u}(t, X^u)) dt \right]. \quad (1.4)$$

and therefore it is sufficient to consider drifts of the form  $\beta_X$  as long as one is interested in the minimizing problem  $V_\epsilon(P_0, P_1)$ .

When  $L$  depends only on  $u$ , problem  $V_\epsilon$  has a counterpart in the deterministic setting. this counterpart has been intensively studied since it is the Monge-Kantorovich problem (for a complete list of references we refer the reader to [11] and [13])

$$\mathcal{T}(P_0, P_1) := \inf \left\{ E \left[ \int_0^1 \ell \left( \frac{d\phi(t)}{dt} \right) dt \right] \middle| P\phi(t)^{-1} = P_t (t = 0, 1), \right. \\ \left. t \mapsto \phi(t) \text{ is absolutely continuous} \right\}. \quad (1.5)$$

Actually the most usual (and better known) form of the Monge-Kantorovich problem is

$$T(P_0, P_1) := \inf \left\{ E(L(Y - X)); X \sim P_0, Y \sim P_1 \right\} \quad (1.6)$$

where  $X \sim P_0$  (resp.  $Y \sim P_1$ ) means that the law of  $X$  (resp.  $Y$ ) is  $P_0$  (resp.  $P_1$ ). It is not difficult to show that  $T(P_0, P_1) = \mathcal{T}(P_0, P_1)$ . In the quadratic case, that is when  $L(t, x, u) = \frac{1}{2}|u|^2$ , the Monge-Kantorovich problem has received much attention, in probability as well as in statistics, in particular because  $\sqrt{T(P_0, P_1)}$ , called Wasserstein metric, metrizes convergence in distribution on the set of probability measures on  $\mathbf{R}^d$  with finite second moments. It is not difficult to show that  $T(P_0, P_1) = \mathcal{T}(P_0, P_1)$ . More recently the results obtained by Brenier (cf. [1], [2]) have revived the subject by enlightening its connection with fluid mechanics and geometry.

Duality results play a fundamental role in the study of Monge-Kantorovich problem. There are two duality results. For the sequel the most important for us is the duality result due to Evans ([5]):

$$T(P_0, P_1) = \sup \left\{ \int_{\mathbf{R}^d} \psi(1, x) P_1(dx) - \int_{\mathbf{R}^d} \psi(0, x) P_0(dx) \right\}, \quad (1.7)$$

where the supremum is taken over all continuous viscosity solutions  $\psi$  to the following Hamilton-Jacobi equation:

$$\frac{\partial \psi(t, x)}{\partial t} + \ell^*(D_x \psi(t, x)) = 0 \quad ((t, x) \in (0, 1) \times \mathbf{R}^d) \quad (1.8)$$

(see E Chap. 3). Here  $D_x := (\partial/\partial x_i)_{i=1}^d$  and for  $z \in \mathbf{R}^d$ ,

$$\ell^*(z) := \sup_{u \in \mathbf{R}^d} \{ \langle z, u \rangle - \ell(u) \}$$

and  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbf{R}^d$ .

The second duality result was chronologically proved before by Kantorovich and implies (1.7) (cf. for instance V):

$$T(P_0, P_1) := \sup \left\{ \int_{\mathbf{R}^d} \psi(y) P_1(dy) + \int_{\mathbf{R}^d} \varphi(x) P_0(dx); \right. \\ \left. (\varphi, \psi) \in L^1(P_0) \times L^1(P_1), \varphi(x) + \psi(y) \leq L(y - x) \right\} \quad (1.9)$$

In the sequel we describe how it is possible to prove a duality theorem for  $V_\epsilon$  in the spirit of (1.7) and describe applications. We will not give all proofs in detail; for detailed proofs we refer the reader to [10].

## 2 Duality Theorem

For simplicity in what follows we restrict to the case when  $L(t, x, u) = L(u)$  (that is  $L$  depends only on  $u$ ). However our main result (duality theorem) and its applications are valid even if  $L$  depends on  $(t, x)$  (cf. [10]). Let us recall that  $P_0$  and  $P_1$  are given Borel probability measures on  $\mathbf{R}^d$ , and  $L(u) : \mathbf{R}^d \mapsto [0, \infty)$  is a measurable and convex function of  $u$ . We moreover assume that

$$V_\epsilon(P_0, P_1) < +\infty \quad (2.1)$$

We will need assumptions on  $L$  which we denote as follows:

(A.1).  $L$  is *superlinear*: for some  $\delta > 1$ ,

$$\liminf_{|u| \rightarrow \infty} \frac{L(u)}{|u|^\delta} > 0.$$

(A.2). (i)  $L \in C^3(\mathbf{R}^d)$ ,

(ii)  $D_u^2 L(u)$  is positive definite for all  $u \in \mathbf{R}^d$ ,

We will look for sufficient conditions for  $V_\epsilon$  to admit a minimizer, unique and/or Markovian and also for a characterization of minimizers. A duality theorem will provide such a characterization (the characterization itself will be obtained in the next section). As already mentioned we focus on the main steps and articulations of the argument.

## 2.1 Existence and uniqueness of a minimizer.

Results about existence and uniqueness are gathered in

**Theorem 2.1** (i)  $V_\epsilon(P_0, P_1)$  admits a minimizer.  
(ii) If assumption (A.1) holds with  $\delta = 2$ ,  $V_\epsilon(P_0, P_1)$  admits a Markovian minimizer  
(iii) If  $L$  is strictly convex and assumption (A.1) holds with  $\delta = 2$ , then  $V_\epsilon(P_0, P_1)$  admits a unique minimizer (which is Markovian from (ii)).

Our tool for the proof of (ii) and (iii) in Theorem 2.1 is the following minimization problem with fixed marginals

$$\underline{V}_\epsilon(P_0, P_1) := \inf \int_0^1 \int_{\mathbf{R}^d} L(b(t, x)) P(t, dx) dt, \quad (2.2)$$

where the infimum is taken over all  $(b(t, x), P(t, dx))$  for which  $P(t, dx)$  ( $0 \leq t \leq 1$ ) are Borel probability measures, on  $\mathbf{R}^d$ , such that  $p(t, x) := P(t, dx)/dx$  exists for all  $t \in (0, 1]$ ,  $P(t, dx) = P_t$  ( $t = 0, 1$ ) and the following Fokker-Planck pde

$$\frac{\partial P(t, dx)}{\partial t} = \frac{\epsilon}{2} \Delta P(t, dx) - \operatorname{div}(b(t, x) P(t, dx)) \quad (2.3)$$

is satisfied. Let us notice that  $\underline{V}_\epsilon$  is a stochastic analog of the problem considered by Benamou and Brenier in [3]. Then

**Proposition 2.1** (cf. [10] Lemma 3.5). Assume (A.1) with  $\delta = 2$  holds. Then  $V_\epsilon(P_0, P_1) = \underline{V}_\epsilon(P_0, P_1)$ .

**Proof of Theorem 2.1.** Proof of (i): Let  $(X_n)$  denote a minimizing sequence of processes in the set  $\mathcal{A}$ ; this means that

$$\lim_{n \rightarrow \infty} E \left[ \int_0^1 L(\beta_{X_n}(t, X_n)) dt \right] = V_\epsilon(P_0, P_1) \quad (2.4)$$

Since  $X_n \in \mathcal{A}$  for all  $n$  and assumption (A.1) holds ( $L$  is superlinear), it follows that the sequence  $(X_n)$  is tight: the sufficient condition for tightness of [14] is satisfied. In particular (A.1) implies that

$$E \left[ \int_0^1 |\beta_{X_n}(t, X_n)|^\delta dt \right] < +\infty \quad (2.5)$$

(with  $\delta > 1$ ). Hence there exists a subsequence  $(X_{n_k})$  weach converges weakly; let us denote its limit by  $(X(t))$ . The process  $X$  belongs to  $\mathcal{A}$ : from [14], Theorem 5, we obtain that  $\frac{1}{\sqrt{\epsilon}}\{X(t) - X(0) - A(t)\}_{t \in [0,1]}$  is a standard Brownian motion and  $\{A(t)\}_{t \in [0,1]}$  is absolutely continuous. Moreover  $(X(t))$  satisfies

$$\begin{aligned} & \lim_{k \rightarrow \infty} E \left[ \int_0^1 L(\beta_{X_{n_k}}(t, X_{n_k})) dt \right] \\ & \geq E \left[ \int_0^1 L \left( \frac{dA(t)}{dt} \right) dt \right]. \end{aligned} \quad (2.6)$$

which implies that it is a minimizer of  $V_\epsilon$ . Inequality (2.6) may be proved following the argument of [9] in the proof of Theorem 1, which is here simplified since  $L$  depends on  $u$  only.

Proof of (ii): we now assume that (A.1) holds with  $\delta = 2$ . Using the same argument as in the proof of (i) one can show that  $\underline{V}_\epsilon(P_0, P_1)$  admits a minimizer. From Proposition 2.1 this minimizer also is a minimizer of  $V_\epsilon$  ( here it is actually sufficient that  $V_\epsilon \geq \underline{V}_\epsilon$ ).

Proof of (ii): we moreover assume that  $L$  is strictly convex. From Proposition (actually it is sufficient that  $V_\epsilon \leq \underline{V}_\epsilon$ ) it is enough to show uniqueness for  $\underline{V}_\epsilon$  (cf. [10] proof of Proposition 2.2 where we use the strict convexity of  $L$  and the linearity of Fokker-Planck pde). Q.E.D.

## 2.2 Duality Theorem.

**Theorem 2.2** *Suppose that (A.1) and (A.2) are satisfied. Then*

$$V_\epsilon(P_0, P_1) = \sup \left\{ \int_{\mathbf{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx) \right\}, \quad (2.7)$$

where the supremum is taken over all classical solutions  $\varphi$ , to the following HJB equation, for which  $\varphi(1, \cdot) \in C_b^\infty(\mathbf{R}^d)$ :

$$\frac{\partial \varphi(t, x)}{\partial t} + \frac{\epsilon}{2} \Delta \varphi(t, x) + H(D_x \varphi(t, x)) = 0 \quad ((t, x) \in (0, 1) \times \mathbf{R}^d) \quad (2.8)$$

**Proof of 2.2** The two main arguments of the proof are:

1. A property of the Legendre transform: on a Banach space if  $f$  is a lower semi continuous function not identically equal to  $+\infty$ , then  $f^{**} = f$  where  $*$  denotes Legendre transform.

2. A representation of the value function of a stochastic control problem (with sufficiently regular terminal cost) by a solution of an Hamilton-Jacobi-Bellman pde.

For point 1., we rely on results of [4] (namely Theorem 2.2.15 and Lemma 3.2.3). To apply these results, one has to prove first that  $P \mapsto V(P_0, P)$  is lower semicontinuous and convex. This is proved in detail in [10] Lemmas 3.1 and 3.2. It follows that

$$V(P_0, P_1) = \sup_{f \in C_b(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P_1(dx) - V_{P_0}^*(f) \right\}, \quad (2.9)$$

where for  $f \in C_b(\mathbf{R}^d)$ ,

$$V_{P_0}^*(f) := \sup_{P \in \mathcal{M}_1(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P(dx) - V(P_0, P) \right\},$$

and  $\mathcal{M}_1(\mathbf{R}^d)$  denotes the complete separable metric space, with a weak topology, of Borel probability measures on  $\mathbf{R}^d$ .

For point 2., we refer the reader to [6]: for  $f \in C_b^\infty(\mathbf{R}^d)$ ,

$$\begin{aligned} V_{P_0}^*(f) &= \sup \left\{ E[f(X(1))] - E \left[ \int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] : \right. \\ &\quad \left. X \in \mathcal{A}, PX(0)^{-1} = P_0 \right\} \\ &= \int_{\mathbf{R}^d} \varphi_f(0, x) P_0(dx), \end{aligned} \quad (2.10)$$

where  $\varphi_f$  denotes the unique classical solution to the HJB equation (2.3) with  $\varphi(1, \cdot) = f(\cdot)$ . Using both identities (2.9) and (2.10), we obtain

$$V_\epsilon(P_0, P_1) \geq \sup_{f \in C_b^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx), \quad (2.11)$$

To prove the converse inequality we have to pass from  $C_b(\mathbf{R}^d)$  to  $C_b^\infty(\mathbf{R}^d)$  with the help of a mollifier sequence. Take  $\Phi \in C_0^\infty([-1, 1]^d; [0, \infty))$  for which  $\int_{\mathbf{R}^d} \Phi(x) dx = 1$ , and for  $\delta > 0$ , and define

$$\Phi_\delta(x) := \delta^{-d} \Phi(x/\delta).$$

For  $f \in C_b(\mathbf{R}^d)$ , we set



$$f_\delta(x) := \int_{\mathbf{R}^d} f(y) \Phi_\delta(x - y) dy. \quad (2.12)$$

Then  $f_\delta \in C_b^\infty(\mathbf{R}^d)$  and

$$\begin{aligned} & \sup_{f \in C_b^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx) \\ & \geq \int_{\mathbf{R}^d} f_\delta(x) P_1(dx) - V_{P_0}^*(f_\delta) \\ & \geq \int_{\mathbf{R}^d} f(x) \Phi_\delta * P_1(dx) - V_{\Phi_\delta * P_0}^*(f). \end{aligned}$$

Indeed, for any  $X \in \mathcal{A}$

$$E[f_\delta(X(1))] = \int_{\mathbf{R}^d} \Phi(z) dz E[f(X(1) - \delta z)] \quad (2.13)$$

Then identity (2.9) implies that

$$\begin{aligned} & \sup_{f \in C_b^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx) \\ & \geq V(\Phi_\delta * P_0, \Phi_\delta * P_1) \end{aligned}$$

It remains to let  $\delta$  go to 0 and use the lower semi-continuity of  $(P, Q) \mapsto V(P, Q)$  proved in [10]. Q.E.D.

### 3 Applications.

#### 3.1 Characterization.

We first recall the following property of Legendre transform which we will use repeatedly: if  $L$  is strictly convex, superlinear (i.e. satisfies (A.1)) and smooth (for instance belongs to  $C^2(\mathbf{R}^d)$ ) then  $L^{**} = L$ ;  $\nabla L : \mathbf{R}^d \rightarrow \mathbf{R}^d$  is a bijection from  $\mathbf{R}^d$  onto itself and  $\nabla H = \nabla L^{-1}$  where  $H = L^*$ . If moreover  $D^2L$  is positive definite,  $H$  is twice differentiable and

$$D^2H(\nabla L(u)) = D^2L(u)^{-1} \quad (3.1)$$

**Theorem 3.1** Suppose that (A.1) and (A.2) hold. Then for any minimizer  $\{X(t)\}_{0 \leq t \leq 1}$  of  $V_\epsilon(P_0, P_1)$ , there exists a sequence of classical solutions  $\{\varphi_n\}_{n \geq 1}$  to the HJB equation (2.8), such that  $\varphi_n(1, \cdot) \in C_b^\infty(\mathbf{R}^d)$  ( $n \geq 1$ ) and that the following holds:

$$\begin{aligned} \beta_X(t, X) &= b_X(t, X(t)) := E[\beta_X(t, X)|(t, X(t))] \\ &= \lim_{n \rightarrow \infty} D_z H(t, X(t); D_x \varphi_n(t, X(t))) \quad dtdPX(\cdot)^{-1} - a.e.. \end{aligned} \quad (3.2)$$

**Proof of Theorem 3.1** From Theorem 2.2 there exists a sequence of classical solutions  $\{\varphi_n\}_{n \geq 1}$  to the HJB equation (2.8), such that  $\varphi_n(1, \cdot) \in C_b^\infty(\mathbf{R}^d)$  ( $n \geq 1$ ) and

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} \varphi_n(1, y) P_1(dy) - \int_{\mathbf{R}^d} \varphi_n(0, x) P_0(dx) = V_\epsilon(P_0, P_1) \quad (3.3)$$

Therefore, for  $X$  a minimizer of  $V_\epsilon$ , it holds

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} \varphi_n(1, y) P_1(dy) - \int_{\mathbf{R}^d} \varphi_n(0, x) P_0(dx) = E \left[ \int_0^1 L(\beta_X(t, X)) dt \right] \quad (3.4)$$

Since  $X(0) \sim P_0$  (resp.  $X(1) \sim P_1$ ) and  $\{\varphi_n\}_{n \geq 1}$  solves the HJB pde (2.8), Ito formula yields

$$\lim_{n \rightarrow \infty} E \int_0^1 \langle \beta_X(t, X), \nabla \varphi_n(t, X(t)) \rangle - L(\beta_X(t, X)) - H(\nabla \varphi_n(t, X(t))) dt = 0 \quad (3.5)$$

Moreover by definition of  $H$  as the Legendre transform of  $L$ , the integrand in (3.5) is positive. Hence the sequence

$$\langle \beta_X(t, X), \nabla \varphi_n(t, X(t)) \rangle - L(\beta_X(t, X)) - H(\nabla \varphi_n(t, X(t))) \quad (3.6)$$

converges to 0 in  $L^1(dtdP)$  and admits a subsequence which converges a.s. For simplicity we still denote this subsequence by  $(\varphi_n)$ . Let  $(t, \omega)$  be such that the sequence  $\langle \beta_X(t, X), \nabla \varphi_n(t, X(t)) \rangle - H(\nabla \varphi_n(t, X(t)))$  converges to  $L(\beta_X) = H^*(\beta_X)$ . The supremum in the definition of

$$H^*(u) = \sup(\langle p, u \rangle - H(p)) \quad (3.7)$$

is attained at  $p^* = \nabla L(u)$ . We therefore obtain that

$$\lim \nabla \varphi_n(t, X(t)) = \nabla L(\beta_X(t, X)) \quad (3.8)$$

or equivalently  $\beta_X(t, X) = \lim \nabla H(\nabla \varphi_n(t, X(t)))$ . Q.E.D.

We would like to show now that a minimizer solves a stochastic equation. We were able to prove such a result under the additional assumption: (A.3).  $D^2L(u)$  is bounded.

The following lemma will be useful below:

**Lemma 3.1** *Let  $L \in C^2(\mathbf{R}^d)$  be strictly convex and superlinear such that*

$$C := \sup\{\langle D^2L(u)z, z \rangle : (u, z) \in \mathbf{R}^d \times \mathbf{R}^d, |z| = 1\} < +\infty \quad (3.9)$$

*Then*

$$\forall (u, z) \in \mathbf{R}^d \times \mathbf{R}^d \quad \|z - \nabla L(u)\|^2 \leq C|L(u) - (\langle u, z \rangle - H(z))| \quad (3.10)$$

**Proof of Lemma 3.1.** By definition of  $H = L^*$ , for all  $(u, z)$ ,  $L(u) - (\langle u, z \rangle - H(z)) \geq 0$ . The assumptions of the lemma ensure that for all  $u$ ,  $u = \nabla H(\nabla L(u))$  and  $H(p) = \langle p, \nabla H(p) \rangle - L(\nabla H(p))$  for all  $p$ . We therefore have

$$L(u) - (\langle u, z \rangle - H(z)) = H(z) - H(\nabla L(u)) - \langle \nabla H(\nabla L(u)), z - \nabla L(u) \rangle \quad (3.11)$$

The conclusion follows from identity (3.1). Q.E.D.

**Theorem 3.2** *Suppose that (A.1) holds with  $\delta = 2$  as well as (A.2) and (A.3). Then for the unique minimizer  $\{X(t)\}_{0 \leq t \leq 1}$  of  $V_\epsilon(P_0, P_1)$ , (1) there exist  $f(\cdot) \in L^1(\mathbf{R}^d, P_1(dx))$  and a  $\sigma[X(s) : 0 \leq s \leq t]$ -continuous semimartingale  $\{Y(t)\}_{0 \leq t \leq 1}$  such that*

$$\{(X(t), Y(t), Z(t) := D_u L(b_X(t, X(t))))\}_{0 \leq t \leq 1}$$

*satisfies the following FBSDE in a weak sense: for  $t \in [0, 1]$ ,*

$$\begin{aligned} X(t) &= X(0) + \int_0^t D_z H(Z(s)) ds + \sqrt{\epsilon} W(t), \\ Y(t) &= f(X(1)) - \int_t^1 L(D_z H(Z(s))) ds \\ &\quad - \int_t^1 \langle Z(s), dW(s) \rangle. \end{aligned} \quad (3.12)$$

(2) there exist  $f_0(\cdot) \in L^1(\mathbf{R}^d, P_0(dx))$  and  $\varphi(\cdot, \cdot) \in L^1([0, 1] \times \mathbf{R}^d, P((t, X(t)) \in dt dx))$  such that  $Y(0) = f_0(X(0))$  and such that

$$Y(t) - Y(0) = \varphi(t, X(t)) - \varphi(0, X(0)) \quad dtdPX(\cdot)^{-1} - a.e., \quad (3.13)$$

that is,  $Y(t)$  is a continuous version of  $\varphi(t, X(t)) - \varphi(0, X(0)) + f_0(X(0))$ .

**Proof of Theorem 3.2** Let  $(\varphi_n)$  be a sequence satisfying the same conditions as in the proof of Theorem 3.1 and  $X$  a minimizer of  $V_\epsilon$ . From Ito formula,

$$\begin{aligned} & \varphi_n(t, X(t)) - \varphi_n(0, X(0)) \\ &= \int_0^t \{ \langle b_X(s, X(s)), D_x \varphi_n(s, X(s)) \rangle - H(D_x \varphi_n(s, X(s))) \} ds \\ & \quad + \int_0^t \langle D_x \varphi_n(s, X(s)), \sqrt{\epsilon} dW(s) \rangle. \end{aligned} \quad (3.14)$$

We first consider convergence of the martingale part. By Doob's inequality

$$\begin{aligned} & E\left(\sup_{0 \leq t \leq 1} \left| \int_0^t \langle D_x \varphi_n(s, X(s)) - D_u L(b_X(s, X(s))), dW(s) \rangle \right|^2\right) \\ & \leq 4E\left(\int_0^1 |D_x \varphi_n(s, X(s)) - D_u L(b_X(s, X(s)))|^2 ds\right) \end{aligned} \quad (3.15)$$

By Lemma 3.1 it follows that

$$\begin{aligned} & E\left(\sup_{0 \leq t \leq 1} \left| \int_0^t \langle D_x \varphi_n(s, X(s)) - D_u L(b_X(s, X(s))), dW(s) \rangle \right|^2\right) \\ & \leq 4CE\left(\int_0^1 |L(b_X(s, X(s))) - (\langle b_X(s, X(s)), D_x \varphi_n(s, X(s)) \rangle \right. \\ & \quad \left. - H(D_x \varphi_n(s, X(s))))| ds\right) \end{aligned}$$

which converges to 0 by Theorem 3.1. This theorem also implies that

$$\int_0^t \{ \langle b_X(s, X(s)), D_x \varphi_n(s, X(s)) \rangle - H(D_x \varphi_n(s, X(s))) \} ds \quad (3.16)$$

converges in  $L^1$  to  $\int_0^1 L(b_X(s, X(s))) ds$ . We therefore obtain that  $\varphi_n(1, y) - \varphi_n(0, x)$  and  $\varphi_n(t, y) - \varphi_n(0, x)$  are convergent in  $L^1(\mathbf{R}^d \times \mathbf{R}^d, P((X(0), X(1)) \in$

$dx dy$ ) and  $L^1(\mathbf{R}^d \times [0, 1] \times \mathbf{R}^d, P((X(0), (t, X(t))) \in dx dt dy))$ , respectively. The question is whether the limit is still of the separable form  $\psi(1, y) - \psi(0, x)$  and  $\psi(t, y) - \psi(0, x)$  respectively. From [12] this is indeed the case provided that the law of  $(X(0), X(1))$  (resp.  $(X(0), X(t))$ ) is absolutely continuous with respect to  $P_0(dx)P_1(dy)$  ( resp.  $P_0(dx)P_t(dy)$ ) where  $P_t$  denotes the law of  $X_t$ . These conditions are satisfied here since (A.1) holds with  $\delta = 2$  and consequently the process  $X$  has finite entropy w.r.t. the Wiener measure on  $C(\mathbf{R}^d)$  with initial law  $P_0$ . Hence, from [12], Prop. 2, there exist  $f \in L^1(\mathbf{R}^d, P_1(dx))$ ,  $f_0 \in L^1(\mathbf{R}^d, P_0(dx))$ ,  $\varphi_0 \in L^1(\mathbf{R}^d, P_0(dx))$  and  $\varphi \in L^1([0, 1] \times \mathbf{R}^d, P((t, X(t)) \in dt dy))$  such that

$$\lim_{n \rightarrow \infty} E[|\varphi_n(1, X(1)) - \varphi_n(0, X(0)) - \{f(X(1)) - f_0(X(0))\}|] = 0, \quad (3.17)$$

and

$$\lim_{n \rightarrow \infty} E\left[\int_0^1 |\varphi_n(t, X(t)) - \varphi_n(0, X(0)) - \{\varphi(t, X(t)) - \varphi_0(X(0))\}| dt\right] = 0. \quad (3.18)$$

It is easy to check that  $(Y(t))$  defined by

$$\begin{aligned} Y(t) &:= f_0(X(0)) + \int_0^t L(s, X(s); b_X(s, X(s))) ds \\ &\quad + \int_0^t \langle D_u L(s, X(s); b_X(s, X(s))), dW(s) \rangle. \end{aligned} \quad (3.19)$$

satisfies the statement of Theorem 3.2. Q.E.D.

## References

- [1] Brenier, Y., *Décomposition polaire et réarrangement monotone des champs de vecteurs*. C. R. Acad. Sci. Paris, Série I, 305, pp. 805- 808 (1987)
- [2] Brenier, Y., *Extended Monge-Kantorovich theory* Proc. of the CIME course 2001 " Optimal Transportation and Applications", Martina France (Italy) Lecture Notes in Mathematics n. 1813, p. 91-121, (2003) Springer, Berlin.

- [3] Brenier, Y., Benamou, J.D., A numerical method for the optimal mass transport problem and related problems, in: Caffarelli, L.A., Milman, M. (Eds.), *Monge Ampre equation: applications to geometry and optimization*, Proc. of the NSF-CBMS Conference, Deerfield Beach, FL 1997, Contemporary Mathematics 226, Amer. Math. Soc., Providence, RI, pp. 1-11 (1999).
- [4] Deuschel, J.D., Stroock, D.W., . *Large deviations*, Pure and Applied Mathematics Vol. 137, Academic Press, Inc., Boston, MA. 1989
- [5] Evans, L.C., 1998. *Partial differential equations*, Graduate Studies in Mathematics Vol. 19, Amer. Math. Soc., Providence, RI, USA. 1998.
- [6] Fleming, W.H., Soner, H.M., . *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo. 1993
- [7] Liptser, R.S., Shiryaev, A.N., *Statistics of random processes I*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo (1977)
- [8] Mikami, T., *Variational processes from the weak forward equation*. Commun. Math. Phys. 135, 19-40 (1990)
- [9] Mikami, T *Optimal control for absolutely continuous stochastic processes and the mass transportation problem*. Elect. Comm. in Probab. 7, 199-213 (2002)
- [10] Mikami, T., Thieullen, M., *Duality Theorem for Stochastic Optimal Control Problem*. Hokkaido University Preprint Series in Mathematics , n. 652, June 2004. Submitted to Stoch. Proc. Appl.
- [11] Rachev, S.T., Rüschendorf, L., *Mass transportation problems*, Vol. I: Theory, Vol. II: Application, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo. 1998.
- [12] Rüschendorf, L., Thomsen, W., *Note on the Schrödinger equation and I-projections*. Statist. Probab. Lett. 17, 369-375 (1993).
- [13] Villani, C., *Topics in Optimal Transportation*, Graduate Studies in Mathematics Vol. 58, Amer. Math. Soc., Providence, RI. 2003

- [14] Zheng, W.A., *Tightness results for laws of diffusion processes application to stochastic mechanics*. Ann. Inst. Henri Poincaré 21, pp. 103-124 (1985)